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On Parametrization in Modelling Behavioral Heterogeneity

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Topics in Mathematical Economics

The participants in a September 2002 Workshop on *Topics in Mathematical Economics* in honor of Birgit Grodal decided to have a series of papers appear on Birgit Grodal's 60'th birthday, June 24, 2003.

The Institute of Economics suggested that the papers became Discussion Papers from the Institute.

The editor of *Economic Theory* offered to consider the papers for a special Festschrift issue of the journal with Karl Vind as Guest Editor.

This paper is one of the many papers sent to the Discussion Paper series.

Most of these papers will later also be published in a special issue of *Economic Theory*.

Tillykke Birgit

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On Parametrization in Modelling Behavioral Heterogeneity

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Abstract

In order to model the concept of behavioral heterogeneity, some authors have employed the following approach. By using affine transformations of the price space, a consumption sector is defined as the set of all transformed demand functions of a given "generating" demand function. This leads to a parametrization of the objects by elements of a finite dimensional Euclidian space. Increasing dispersedness of the parameter distribution is then interpreted as increasing behavioral heterogeneity of the consumption sector.

In this paper, we show that such an interpretation is not valid if the generating demand function satisfies a mild regularity condition. Due to the specific parametrization, *increasing* heterogeneity of the parameters leads to *decreasing* heterogeneity of the demand functions. More precisely, we prove that they become concentrated nearby a finite set of Cobb-Douglas demand functions.

Keywords: Aggregate Demand, Behavioral Heterogeneity, Parametrization

JEL Classification System: D11, D30, D50

1 Introduction

Following the seminal contribution by Grandmont (1992), several attempts have been made to relate desirable structural properties of aggregate demand to the idea of behavioral heterogeneity (see e.g. Quah (1997), Kneip (1999), and, recently, Giraud and Maret (2001)). In order to discuss such a relationship more precisely, let us consider the following framework.

Given a fixed income, the behavior of a consumer is described by a demand function f that assigns to every price vector $p \in \mathbb{R}^l_{++}$ a commodity bundle $f(p) \in \mathbb{R}^l_+$. Assuming the set $\mathcal{F}$ of all demand functions to be equipped with a suitable topology, a consumption sector is then defined by a Borel probability measure τ on $\mathcal{F}$. The mean (or aggregate) demand of the consumption sector at p is

$$F(p) = \int_{\mathcal{F}} f(p) d\tau.$$

As a part of a well-behaved general equilibrium model, F should have properties that guarantee uniqueness and stability of equilibrium. For that, it is sufficient that F is, at least approximately, of Cobb-Douglas type. Actually, most of the mentioned contributions pursue this aim. But how is it related to a concept of behavioral heterogeneity?

Intuitively, one would tend to speak of behavioral heterogeneity, if the support of the distribution τ is "large" and, furthermore, τ is not concentrated on a "small" subset of $\mathcal{F}$ but puts approximately the same mass on sets of "equal size". However, it is difficult to give a precise meaning of these notions in case of the infinite dimensional function space $\mathcal{F}$. Therefore, some authors, like Grandmont (1992) and Quah (1997), consider a parametrization of the demand functions by a finite dimensional space, e.g. a subset C of $\mathbb{R}^n$. It is defined by a mapping T from C into $\mathcal{F}$ such that a distribution μ on C induces a consumption sector τ as the image measure of μ with respect to T.

Obviously, a parameter distribution μ that is dispersed - a concept which is well defined for μ - does not necessarily imply that the induced consumption sector τ displays behavioral heterogeneity. As put by Quah (1997), "since dispersion in this paper (and Grandmont's) is a condition imposed on the parameter distribution, it is not clear if it is related to heterogeneity (in some meaningful sense) of the distribution of demand functions."

The intention of this paper is to show, by using Grandmont's approach as an example, that this question has to be answered negatively. More precisely, we consider a sequence of measures (μ_n) on the parameter space $\mathbb{R}^l$ which are increasingly dispersed in the sense that their densities become increasingly flat. Moreover, T is defined by applying affine transformations of the price space to a given "generating" demand function that is linear in income and satisfies a mild regularity condition at the boundary of the price space. Then, it turns out, that the induced measures τ_n on $\mathcal{F}$ become increasingly concentrated close to a finite set of Cobb-Dougas demand functions. Consequently, it is no surprise that aggregate demand will be approximately of Cobb-Douglas type. Put differently, it is increasing behavioral similarity which is responsible for the nice structural properties of aggregate demand.

The message of this paper is that every parametrization contains an inherent danger of misinterpretation. In our case, easily interpretable assumptions on the parameter distribution μ imply quite unintentional properties of the induced distribution τ of demand functions.

The following section provides an example that conveys the basic idea in a simple way. Section 3 presents the general results while proofs are gathered in Section 4. Some additional remarks are offered in the concluding Section 5.

2 An Example

Let $\mathbb{R}^2$ be the initial parameter space and assume the parameters $\alpha = (\alpha_1, \alpha_2)$ to be distributed according to the two-dimensional, uncorrelated, and symmetric normal distribution μ_σ, defined by the density

$$\rho_\sigma(\alpha_1, \alpha_2) = \frac{1}{2\pi\sigma^2} \exp(-\frac{\alpha_1^2 + \alpha_2^2}{2\sigma^2}), \quad \sigma > 0.$$

Obviously, increasing heterogeneity of the α-parameters (or increasing dispersion of the distribution μ_σ) is described by an increasing flatness of the density ρ_σ, i.e. by $\sigma \to \infty$.

Consider now the composite mapping

$$\alpha \mapsto a = \exp(\alpha) := (\exp(\alpha_1), \exp(\alpha_2)) \mapsto \frac{\exp(\alpha)}{\exp(\alpha_1) + \exp(\alpha_2)}$$

from $\mathbb{R}^2$ into the unit simplex $\Delta = \{a \in \mathbb{R}^2_+ \mid a_1 + a_2 = 1\}$.

The inverse image of $(\delta, 1-\delta) \in \Delta$, $\delta > 0$, with respect to this mapping is the straight line

$$G_\delta = \{(\ln\delta, \ln(1-\delta)) + (\ln\lambda, \ln\lambda) \mid \lambda > 0\} \subseteq \mathbb{R}^2.$$

Similarly, we denote the inverse images of the sets $S_1 := \{(\delta', 1-\delta') \,|\, \delta' > 1-\delta\}, S_2 := \{(\delta', 1-\delta') \,|\, \delta' < \delta\}$, and $S_0 := \{(\delta', 1-\delta') \,|\, \delta < \delta' < 1-\delta\}$ by A_1, A_2, and A_0. These sets are depicted in Figures 1 and 2 that illustrate the relationship between the α-parameter space $\mathbb{R}^2$, the a-parameter space $\mathbb{R}^2_{++}$, and the open unit simplex S as the δ-parameter space.

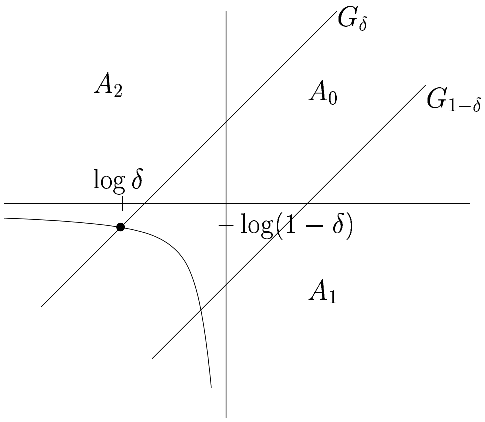

Figure 1

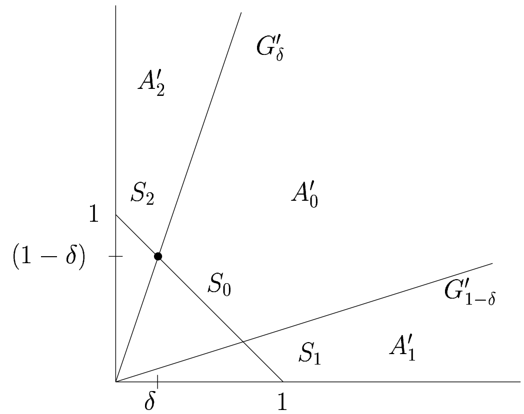

Figure 2

The sets $A'_2, G'_\delta, A'_0, G'_{1-\delta}$, and A'_1 in Figure 2 correspond to the sets $A_2, G_\delta, A_0, G_{1-\delta}$, and A_1 in Figure 1 and the hyperbola-like graph in Fig. 1 corresponds to S with respect to the transformation $\alpha \mapsto \exp(\alpha)$.

Any measure μ_σ on $\mathbb{R}^2$ induces an image measure ν_σ on Δ which is defined by

$$\nu_\sigma(D) = \mu_\sigma(\{\alpha \in \mathbb{R}^2 \,|\, \frac{\exp(\alpha)}{\exp(\alpha_1)+\exp(\alpha_2)} \in D\}).$$

By the symmetry of μ_σ, it is easy to see that, for fixed δ between 0 and $\frac{1}{2}$, one obtains for $\sigma \to \infty$

$$\begin{aligned} \nu_\sigma(S_1) &= \mu_\sigma(A_1) \to \frac{1}{2} \\ \nu_\sigma(S_0) &= \mu_\sigma(A_0) \to 0 \\ \nu_\sigma(S_2) &= \mu_\sigma(A_2) \to \frac{1}{2} \;\; . \end{aligned}$$

It follows that $\lim_{\sigma\to\infty} \nu_\sigma = \nu_\infty$ with respect to the weak topology, where the limit measure ν_∞ is given by $\nu_\infty(\{(1,0)\}) = \nu_\infty(\{(0,1)\}) = \frac{1}{2}$ and $\nu_\infty(S) = 0$.

The interpretation of this result is obvious: Increasing flatness of ρ_σ, i.e. increasing dispersion of the α-parameters, leads to an increasing concentration of the a-parameters close to the axes (resp. the δ-parameters close to the boundary points of S).

So far we have only discussed the relationship between the distributions of the different parameters. Now, we turn to our objects of interest which are demand functions for two commodities.

Given a fixed income equal to one, these are functions f from the price space $\mathbb{R}^2_{++}$ into the consumption set $\mathbb{R}^2_{+}$. They are parametrized by $\alpha \in \mathbb{R}^2$ according to the definition[1]

$$f^a(p) := a \otimes f(a \otimes p)\,,\ a = \exp(\alpha)$$

where f is a given "generating" demand function which is continuous, (-1)-homogeneous[2], and satisfies

(C) The limits $\lim_{p\to(1,0)} p_1 f_1(p) =: \gamma_1$ and $\lim_{p\to(0,1)} p_2 f_2(p) =: \gamma_2$ exist.

This assumption excludes that budget shares wiggle at prices close to the boundary of the price space and is reasonable from an economic viewpoint.

By the homogeneity property of f, it follows that $f^{\exp(\alpha)} = f^a$, where $a = \exp(\alpha)/(\exp(\alpha_1) + exp(\alpha_2))$, i.e. the essential parametrization is given by $f^{(\delta,1-\delta)}$ for $0 < \delta < 1$.

Of course, we want to know how these functions look like if δ is close to 0 or 1 since increasing heterogeneity of the α-parameters puts most mass on those δ.

By definition, we obtain the pointwise convergence

$$\begin{aligned} p_1 f_1^{(\delta,1-\delta)}(p) &= \delta p_1 f_1(\delta p_1, (1-\delta)p_2) \\ &= \frac{\delta p_1}{\delta p_1 + (1-\delta)p_2} f_1\left(\frac{\delta p_1}{\delta p_1 + (1-\delta)p_2}, \frac{(1-\delta)p_2}{\delta p_1 + (1-\delta)p_2}\right) \underset{\delta\to 1}{\longrightarrow} \gamma_1 \end{aligned}$$

Analogously, $p_2 f_2^{(\delta,1-\delta)}(p) \underset{\delta\to 0}{\longrightarrow} \gamma_2$.

It can be easily shown that both convergences are uniform on compact subsets of the price space $\mathbb{R}^2_{++}$. Hence, with respect to this topology, $\lim_{\delta\to 1} f^{(\delta,1-\delta)} = \bar{f}^1$ and $\lim_{\delta\to 0} f^{(\delta,1-\delta)} = \bar{f}^2$, where $\bar{f}^1(p) = \left(\frac{\gamma_1}{p_1}, \frac{1-\gamma_1}{p_2}\right)$ and $\bar{f}^2(p) = \left(\frac{1-\gamma_2}{p_1}, \frac{\gamma_2}{p_2}\right)$ are two Cobb-Douglas demand functions.

It follows that for an arbitrary open neighborhood $\mathcal{O}$ of $\bar{f}^1$ and $\bar{f}^2$ there exists δ with $0 < \delta < \frac{1}{2}$ and $f^{(\delta',1-\delta')} \in \mathcal{O}$ for $(\delta', 1-\delta') \in S_1 \cup S_2$.

[1] $x \otimes y = (x_1 y_1, x_2 y_2)$

[2] This corresponds to a demand function with variable income that is 0-homogeneous and linear w.r.t. income.

For the image measure τ_σ of ν_σ with respect to the mapping $s \mapsto f^s$ which is defined by

$$\tau_\sigma(F) = \nu_\sigma(\{s \in S \mid f^s \in F\})$$

on the space $\mathcal{F}$ of all continuous demand functions we thus obtain

$$\tau_\sigma(\mathcal{O}) \geq \nu_\sigma(S_1 \cup S_2) = \nu_\sigma(S_1) + \nu_\sigma(S_2)$$

and, consequently, $\lim_{\sigma \to \infty} \tau_\sigma(\mathcal{O}) = 1$.

This means that an increasing heterogeneity of the α-parameters leads to an increasing concentration of the parametrized objects close to the two Cobb-Douglas function $\bar{f}^1$ and $\bar{f}^2$. Hence, it is no surprise that aggregate demand is approximately of Cobb-Douglas type if the density ρ_σ becomes extremely flat.

3 General Results

We consider consumers that are described by a continuous demand function $f : \mathbb{P}^l \times \mathbb{P} \longrightarrow \mathbb{R}^l_+$, where $\mathbb{P}$ denotes the set of positive real numbers, such that the following conditions are satisfied for all $p \in \mathbb{P}^l$ and all $b, \lambda \in \mathbb{P}$:

(1) $f(\lambda p, \lambda b) = f(p, b)$ (homogeneity)
(2) $pf(p, b) = b$ (budget identity)
(3) $f(p, \lambda b) = \lambda f(p, b)$ (linearity in income).

It is straightforward to check that the set $\mathcal{F}$ of all these demand functions can be equivalently described by their budget share functions. More precisely, to any $f \in \mathcal{F}$ one can assign a continuous function $w : S \longrightarrow \Delta$, where S (resp. Δ) denotes the open (resp. closed) unit simplex in $\mathbb{R}^l$, such that the assignment defines a bijection between $\mathcal{F}$ and the set $\mathcal{W} = C(S, \Delta)$ given by $f \mapsto w(f), w(f)(p) = p \otimes f(p, 1)$.

In the sequel, a consumer is given by $w \in \mathcal{W}$, where $\mathcal{W}$ is endowed with the topology of uniform convergence on compact subsets of S. Consequently, a *consumption sector* is described by a (probability) measure τ on the Borel space $(\mathcal{W}, \mathcal{B}(\mathcal{W}))$.

For any $a \in \mathbb{P}^l$, the *a-transform* of $w \in \mathcal{W}$ is defined by

$$w^a(p) := w(\frac{a \otimes p}{|a \otimes p|}),$$

where $|\cdot|$ denotes the sum-norm on $\mathbb{R}^l$. This transformation corresponds to the a-transform of a demand function f which, for any $a \in \mathbb{P}^l$, is given by

$$f^a(p,b) := a \otimes f(a \otimes p, b).$$

Originally, Grandmont (1992) used $\mathbb{R}^l$ as a parameter space, i.e. any $a \in \mathbb{P}^l$ is described by $\exp(\alpha)$ for $\alpha \in \mathbb{R}^l$. Accordingly, the α-transform of $w \in \mathcal{W}$ is defined by $w^{\exp(\alpha)}$. Given any fixed ("generating") $w \in \mathcal{W}$, a measure μ on $\mathbb{R}^l$ representing the distribution of the parameters α gives rise to a consumption sector $\tau(\mu)$ which is defined as the image measure of μ with respect to the mapping $\alpha \mapsto w^{\exp(\alpha)}$, i.e.

$$\tau(\mu)(W) = \mu(\{\alpha \in \mathbb{R}^l \mid w^{\exp(\alpha)} \in W\}), \; W \in \mathcal{B}(\mathcal{W}).$$

The mapping $\alpha \mapsto w^{\exp(\alpha)}$ can be decomposed into the two mappings $\alpha \mapsto \exp(\alpha) / |\exp(\alpha)|$ from $\mathbb{R}^l$ into S and $s \mapsto w^s$ from S into $\mathcal{W}$. We denote by $\nu(\mu)$ the image measure of μ with respect to the first mapping and by $\tau(\nu)$ the image measure of a measure ν on S with respect to the second one. Clearly, $\tau(\mu) = \tau(\nu(\mu))$.

Of course, the degree of heterogeneity of a consumption sector can only be judged by looking at $\tau(\mu)$, the measures μ or $\nu(\mu)$ are irrelevant in this respect.

Following Grandmont (1992), we assume that the measure μ on the α-parameter space $\mathbb{R}^l$ is given by a continuously differentiable density ρ. Its *flatness* is described by the number

$$m(\rho) := \max_h \int |\frac{\partial \rho}{\partial \alpha_h}(\alpha)| d\alpha.$$

We are interested in the measure $\tau(\mu)$ for the case of a very flat density ρ, i.e. for $m(\rho)$ very small, since this is interpreted as a highly dispersed (or heterogeneous) distribution of the α-parameters. In order to model such a case, we consider a sequence (ρ_n) of densities with $\lim_{n\to\infty} m(\rho_n) = 0$. The corresponding measures are denoted by μ_n, $\nu_n := \nu(\mu_n)$, and $\tau_n := \tau(\nu_n) = \tau(\mu_n)$.

We assume that generating budget share function w reflects a kind of "regular behavior" at least near the vertices of the price simplex S. This is expressed by the following condition

(C) For every sequence (p^k) in S converging to the i-th unit vector $e^i \in \Delta$, the limit $\lim_{k\to\infty} w(p^k)$ exists and is denoted by $w(e^i)$.

Now we can state the main result as the following

Theorem. Let (ρ_n) be a sequence of densities on $\mathbb{R}^l$ such that $\lim_{n\to\infty} m(\rho_n) = 0$ and assume that $w \in \mathcal{W}$ satisfies the condition (C). Then, for every open neighborhood $\mathcal{O}$ of the (at most) l constant budget share functions $\bar{w}^i(p) \equiv w(e^i)$,

$$\lim_{n\to\infty} \tau_n(\mathcal{O}) = 1.$$

The interpretation of this result is obvious: Given the regularity assumption (C) on the generating budget share function w, *increasing heterogeneity* of the α-parameters implies an *increasing concentration* of the distribution of the associated functions w^α. Put differently, it implies a *decreasing heterogeneity* of the consumption sectors.

The proof of this result rests on the following two propositions that separate the two assumptions of the Theorem.

Proposition 1. If (ρ_n) is a sequence of densities on $\mathbb{R}^l$ such that $\lim_{n\to\infty} m(\rho_n) = 0$, then $\lim_{n\to\infty} \nu_n(U) = 1$ for every open neighborhood U of the vertices of Δ.

Proposition 2. If w satisfies (C) then the mapping $s \mapsto w^s$ can be continuously extended to the vertices of Δ by defining $w^{e_i}(p) = w(e_i)$.

4 Proofs

It will be convenient to separate an essential step in the proof of Proposition 1 from the remaining parts by stating an additional lemma. In order to do that, we first introduce some notation.

Given a vector of direction r in $\mathbb{R}^l$, i.e. $\sum r_h^2 = 1$, the probability distribution μ on $\mathbb{R}^l$, generated by the density function ρ, defines (according to the Theorem of Fubini) on the one-dimensional subspace $<r>$ a marginal measure with density

$$\varphi_r(\xi) = \int_{<r>^\perp} \rho_r(\xi, x')dx',$$

where ρ_r denotes the function ρ written in the transformed coordinates ξ and x' with respect to the subspaces $<r>$ and $<r>^\perp$, respectively. But note that, without further arguments, we cannot exclude the case that $\varphi_r(\xi) = \infty$ for a null-set of ξ-values.

Denote by $\frac{\partial\rho}{\partial r} := r\cdot$ grad(ρ) the directional derivative of ρ in direction r.

Lemma:

$$\text{(i)}\quad \textit{If}\ \ m(\rho) := \max_h \int |\frac{\partial\rho(\alpha)}{\partial\alpha_h}|d\alpha < \infty$$

$$\textit{then}\ \ m(\rho, r) := \int |\frac{\partial\rho(\alpha)}{\partial r}|d\alpha \leq \sqrt{l}\cdot m(\rho)$$

$$\text{(ii)}\quad \textit{If}\ \ m(\rho, r) = \int |\frac{\partial\rho(\alpha)}{\partial r}|d\alpha < \infty$$

$$\textit{then}\ \ ||\varphi_r|| := \sup\{\varphi_r(\xi)\mid \xi\in \mathbb{R}\} \leq \frac{1}{2}m(\rho, r).$$

Proof: Using $\frac{\partial\rho}{\partial r} = r\cdot \text{grad}\,(\rho)$ and $\max\{\sum |r_i| \mid \sum r_i^2 = 1\} = \sqrt{l}$ we obtain (i) by

$$\begin{aligned}\int |\frac{\partial\rho}{\partial r}| &= \int |\sum r_h \frac{\partial\rho}{\partial\alpha_h}| \leq \int \sum |r_h\frac{\partial\rho}{\partial\alpha_h}| = \int\sum |r_h|\cdot|\frac{\partial\rho}{\partial\alpha_h}|\\ &= \sum \big(|r_h|\int |\frac{\partial\rho}{\partial\alpha_h}|\big) \leq \big(\sum |r_h|\big)\cdot\max_h \int |\frac{\partial\rho}{\partial\alpha_h}| \leq \sqrt{l}\cdot m(\rho).\end{aligned}$$

Now we prove (ii). If we knew that for every $\bar\xi$ there are an $\epsilon > 0$ and integrable functions $g, g_1 : \mathbb{R}^{l-1} \to \mathbb{R}_+$ with

$$|\rho_r(\xi, x')| \leq g(x') \quad \text{and} \quad \Big|\frac{\partial\rho_r}{\partial r}(\xi, x')\Big| \leq g_1(x')$$

$$\text{for all } \xi \in [\bar\xi-\epsilon, \bar\xi+\epsilon] \qquad \text{and} \qquad x'\in\mathbb{R}^{l-1}$$

then we would obtain (Dieudonné (1970), Th. 13.8.6) that φ_r is continuously differentiable and

$$\varphi_r'(\bar\xi) = \int_{<r>^\perp} \frac{\partial\rho_r}{\partial\xi}(\bar\xi, x')dx'.$$

Clearly, the needed condition is fulfilled if μ is a product measure with respect to the subspaces $<r>$ and $<r>^\perp$. But in general we have to be more explicit. Define for every natural number k the function

$$\rho_{rk}(\xi, x') := \begin{cases} \rho_r(\xi, x') & \text{if } x'\in[-k,k]^{l-1}\\ 0 & \text{otherwise}\end{cases}.$$

Clearly, for ρ_{rk} the above stated condition is fulfilled, while the discontinuity of $\rho_{rk}(\xi,\cdot)$ on the boundary of the cube $[-k,k]^{l-1}$ does not matter.

Since $\int \varphi_{rk} \leq 1$, there exist sequences $\underline{\xi}_n \to -\infty$ and $\bar{\xi}_n \to +\infty$ with $\varphi_{rk}(\underline{\xi}_n) \to 0$ and $\varphi_{rk}(\bar{\xi}_n) \to 0$ and therefore

$$\varphi_{rk}(\bar{\xi}) = \int_{-\infty}^{\bar{\xi}} \varphi'_{rk}(\xi)d\xi = -\int_{\bar{\xi}}^{\infty} \varphi'_{rk}(\xi)d\xi.$$

Hence,

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} \varphi'_{rk}(\xi)d\xi = \int_{-\infty}^{\infty}\int_{<r>^{\perp}} \frac{\partial \rho_{rk}}{\partial \xi}(\xi, x')dx'd\xi = \int_{\mathbb{R}^l} \frac{\partial \rho_k}{\partial r}(\alpha)d\alpha \\ &= \int_{\frac{\partial \rho}{\partial r}(\alpha)\geq 0} \frac{\partial \rho_k}{\partial r}(\alpha)d\alpha + \int_{\frac{\partial \rho}{\partial r}(\alpha)\leq 0} \frac{\partial \rho_k}{\partial r}(\alpha)d\alpha. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} m(\rho_k, r) &= \int_{\mathbb{R}^l} |\frac{\partial \rho_k}{\partial r}(\alpha)|d\alpha \\ &= \int_{\frac{\partial \rho}{\partial r}(\alpha)\geq 0} \frac{\partial \rho_k}{\partial r}(\alpha)d\alpha - \int_{\frac{\partial \rho}{\partial r}(\alpha)\leq 0} \frac{\partial \rho_k}{\partial r}(\alpha)d\alpha \\ &= 2\int_{\frac{\partial \rho}{\partial r}(\alpha)\geq 0} \frac{\partial \rho_k}{\partial r}(\alpha)d\alpha. \end{aligned}$$

Hence

$$\begin{aligned} \varphi_{rk}(\bar{\xi}) &= \int_{-\infty}^{\bar{\xi}} \varphi'_{rk}(\xi)d\xi \leq \int_{\varphi'_{rk}(\xi)\geq 0} \varphi'_{rk}(\xi)d\xi \\ &\leq \int_{-\infty}^{\infty}\int_{\frac{\partial \rho_r}{\partial \xi}(\xi,x')\geq 0} \frac{\partial \rho_{rk}}{\partial \xi}(\xi, x')dx'd\xi \leq \frac{m(\rho, r)}{2}. \end{aligned}$$

Hence, we have shown

$$\varphi_{rk}(\bar{\xi}) \leq \frac{m(\rho, r)}{2} \quad \text{for all} \quad k \text{ and } \xi.$$

Since $\rho_{rk}, k = 1, 2, \ldots$, is pointwise increasing, we obtain with the Theorem of Lebesque

$$\varphi_r(\bar{\xi}) = \lim_{k\to\infty} \varphi_{rk}(\xi) \leq \frac{m(\rho, r)}{2}.$$

□

Now, we can give the

Proof of Proposition 1:

First, we define for an arbitrary δ with $0 < \delta < \frac{1}{l}$ a special neighborhood U^δ of the unit vectors.

For $i, j \in \{1, \dots, l\}$ such that $i \neq j$, let $S^\delta_{ij} := \{s \in S \mid s_i, s_j \geq \delta\}$.

Define $U^\delta_{ij} := S \setminus S^\delta_{ij}$ and $U^\delta := \bigcap_{i \neq j} U^\delta_{ij}$. It is straightforward to check that

$$U^\delta \subseteq \{s \in S \mid \max_h s_h > 1 - (l-1)\delta\}.$$

Indeed, if $s_i := \max_h s_h \leq 1 - (l-1)\delta$, then we obtain for $s_j := \max_{h \neq i} s_h$ the inequalities

$$s_i \geq s_j \geq \frac{1 - s_i}{l - 1} \geq \delta,$$

i.e. $s \in S^\delta_{ij}$. Hence, $s \notin U^\delta_{ij}$ and, therefore, $s \notin U^\delta$.

The inclusion implies that for any open neighborhood U of the unit vectors there exists $\delta > 0$ such that $U^\delta \subseteq U$. It follows that

$$\nu_n(U) \geq \nu_n(U^\delta) = \nu_n(S \setminus \bigcup_{i \neq j} S^\delta_{ij}) = 1 - \nu_n(\bigcup_{i \neq j} S^\delta_{ij}) \geq 1 - \sum_{i \neq j} \nu_n(S^\delta_{ij}).$$

Consequently, $\lim_{n \to \infty} \nu_n(U) \geq 1 - \sum_{i \neq j} \lim_{n \to \infty} \nu_n(S^\delta_{ij})$. Thus, the claim of Proposition 1 is proved if we can show that $\lim_{n \to \infty} \nu_n(S^\delta_{ij}) = 0$ for all $i \neq j$ and the given δ.

By definition,

$$\nu_n(S^\delta_{ij}) = \mu_n(\{\alpha \in \mathbb{R}^l \mid \frac{\exp(\alpha)}{|\exp(\alpha)|} \in S^\delta_{ij}\}) = \mu_n(\{\ln \lambda s \mid s \in S^\delta_{ij}, \lambda > 0\}).$$

Consider the direction $r^{ij} := \frac{1}{\sqrt{2}}(e^i - e^j)$ in $\mathbb{R}^l$. The length of the projection of $\ln \lambda s$ onto $< r^{ij} >$ is then given by

$$|r^{ij} \cdot \ln \lambda s| = \frac{1}{\sqrt{2}} \cdot |\ln s_i - \ln s_j| \leq \frac{1}{\sqrt{2}} |\ln \delta|$$

if $s \in S^\delta_{ij}$. Consequently,

$$\{\ln \lambda s \mid s \in S^\delta_{ij}, \lambda > 0\} \subseteq \{\beta r^{ij} + \gamma \mid \ |\beta| \leq \tfrac{1}{\sqrt{2}} |\ln \delta|,\ \gamma \in < r^{ij} >^\perp\} =: B^\delta_{ij}.$$

It remains to show that $\lim_{n\to\infty} \mu_n(B^\delta_{ij}) = 0$. Denoting by φ_n the density of the marginal distribution of μ_n with respect to $< r^{ij} >$, we obtain

$$\mu_n(B^\delta_{ij}) = \int\limits_{|\beta|\le \frac{1}{\sqrt{2}}|\ln\delta|} \varphi_n(\beta)d\beta \le \sqrt{2}\cdot|\ln\delta|\cdot||\varphi_n||.$$

By the Lemma, $||\varphi_n|| \le \frac{1}{2}\sqrt{l}\cdot m(\rho_n)$. Hence, $\mu_n(B^\delta_{ij}) \le \frac{1}{\sqrt{2}}\sqrt{l}\cdot|\ln\delta|\cdot m(\rho_n)$. Since, by assumption, $\lim_{n\to\infty} m(\rho_n) = 0$ it follows that $\lim_{n\to\infty} \mu_n(B^\delta_{ij}) = 0$. □

Proof of Proposition 2:

Let (s^k) be a sequence in S such that $\lim_{k\to\infty} s^k = e^i$. It has to be shown that the sequence (w^{s^k}) converges uniformly on compact subsets of S to the constant function $\bar{w}^i(p) = w(e^i)$.

We have to prove that for any given compact set $K \subseteq S$ and any $\epsilon > 0$ there exists a number $k(\epsilon)$ such that

$$\left| w(\frac{s^k \otimes p}{|s^k \otimes p|}) - w(e^i) \right| < \epsilon \text{ for all } p \in K \text{ and } k \ge k(\epsilon).$$

By assumption (C), there exists $\delta(\epsilon) > 0$ such that $|w(q) - w(e^i)| < \epsilon$ for all $q \in S$ with $|q - e^i| < \delta(\epsilon)$. Hence, it remains to show that for every $\delta > 0$ there exists $k(\delta)$ such that

$$\left| \frac{s^k \otimes p}{|s^k \otimes p|} - e^i \right| < \delta \quad \text{for } k \ge k(\delta) \text{ and } p \in K.$$

Put differently, it has to be proved that $(q^k) = ((s^k \otimes p)/|s^k \otimes p|)$ converges to e^i uniformly on an arbitrary compact subset K of S.

Without loss of generality, let $K = \{p \in S \mid \min_h p_h \ge \sigma\}$ with $\sigma > 0$. Consider first the i-th component of (q^k). We obtain

$$\begin{aligned} 1 \quad \ge \quad & q^k_i = \frac{s^k_i p_i}{\sum_j s^k_j p_j} = \frac{s^k_i p_i}{s^k_i p_i + \sum_{i\ne j} s^k_j p_j} \\ \ge \quad & \frac{s^k_i p_i}{s^k_i p_i + \sum_{j\ne i} s^k_j} = \frac{s^k_i}{s^k_i + (1-s^k_i)/p_i} \ge \frac{s^k_i}{s^k_i + (1-s^k_i)/\sigma}. \end{aligned}$$

If $s^k \to e^i$, i.e. $s^k_i \to 1$, the last term converges to 1. Hence, (q^k_i) converges to 1 uniformly on K.

For all components j $(j \neq i)$ of (q^k) we obtain

$$0 \leq \frac{s_j^k p_j}{\sum_h s_h^k p_h} \leq \frac{s_j^k}{\sum_h s_h^k \sigma} = \frac{s_j^k}{\sigma \sum_h s_h^k} = \frac{s_j^k}{\sigma}.$$

Since $s_j^k \to 0$ for $j \neq i$, the last term converges to 0. Consequently, (q_j^k) converges to 0 uniformly on K. □

Proof of the Theorem:

Under the assumptions of the Theorem, let $\mathcal{O}$ be an open neighborhood of the constant bugdet share functions $\bar{w}^i$. By Proposition 2, there exists an open neighborhood U of the unit vectors $e^1, \ldots, e^l$ such that $\{w^s \mid s \in U\} \subseteq \mathcal{O}$. This implies

$$\tau_n(\mathcal{O}) \geq \tau_n(\{w^s \mid s \in U\}) = \nu_n(U).$$

By Proposition 1, $\lim_{n\to\infty} \nu_n(U) = 1$ which implies $\lim_{n\to\infty} \tau_n(\mathcal{O}) = 1$. □

5 Concluding Remarks

Recently, it has been argued by Quah (2001) that Grandmont's model *can be* a model of demand heterogeneity. We do not deny this possibility. Indeed, our conlusion rests on the condition (C). However, what is the economic justification for *rejecting* this condition?

A similar problem arises in the contribution by Giraud and Maret (2001) who pursue a slightly different approach. Instead of deriving approximate results for aggregate demand, they try to model a "perfectly heterogeneous" consumption sector by claiming the existence of a "uniform" distribution on the set of all affine transformations of a generating budget share function such that aggregate demand is exactly of Cobb-Douglas type.

Now, assume that one component of the generating function has a partial derivative with respect to the price of some other commodity that is either everywhere positive or everywhere negative. Since affine transformations preserve the sign of the derivatives, the corresponding partial derivative of the aggregate budget share function cannot be zero as required by a Cobb-Douglas type demand. Consequently, such "nice" generating functions have to be excluded in order that their claim is true. Again, what is the economic rationale for this exclusion?

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