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Jens Leth Hougaard, Bezalel Peleg and Lars Peter Østerdal

Studiestræde 6, DK-1455 Copenhagen K., Denmark
Tel. +45 35 32 30 82 - Fax +45 35 32 30 00
http://www.econ.ku.dk

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Jens Leth Hougaard
University of Copenhagen

Bezalel Peleg
The Hebrew University of Jerusalem

Lars Peter Østerdal
University of Copenhagen

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Abstract

This paper considers generalized Lorenz-maximal solutions in the core of a convex TU-game and demonstrates that such solutions satisfy coalitional monotonicity and population monotonicity.

Keywords: Convex games, Core solutions, Generalized Lorenz-maxima, Coalitional monotonicity, Population monotonicity.

Correspondence: Jens Leth Hougaard, Institute of Economics, University of Copenhagen, Studiestraede 6, 1455 Copenhagen K, Denmark. E-mail: Jens.Leth.Hougaard@econ.ku.dk

1 Introduction

A well known result in Young (1985) states that no core solution concept with respect to transferable utility (TU) games satisfies coalitional monotonicity (in the sense that if the worth of a coalition increases *ceteris paribus*, no member of the coalition should receive a decreasing payoff).

This trade-off between coalitional stability, in terms of core constraints, and coalitional monotonicity is unfortunate as illustrated very clearly for the case of cost sharing games.

When cost sharing rules are used in organizations it is important that they are coalitional monotonic since otherwise (some) agents do not have incentives to reduce costs, cf. e.g. Shubik (1962) and Young (1994). Moreover, coalitional stability with respect to cost sharing games is equivalent to the well known stand-alone cost principle stating that no coalition will accept to pay more than their stand-alone cost. Hence, a trade-off between monotonicity and coalitional stability seems difficult to accept.

Moreover, it has long been recognized that population monotonicity is another important monotonicity property, see e.g. Thomson (1995). A solution fails to be population monotonic, if (some) agents have incentives to block the introduction of new agents. In other words, if insiders are able to block the inclusion of new agents, population monotonicity ensures that no agents will exploit such a power irrespectively of the contribution of the outsiders.

In the particular case of convex games, it is well known that (both types of) monotonicity and stability can be compatible as demonstrated by the Shapley value, see Shapley (1971), Sprumont (1990) and Rosenthal (1990). On the other hand, certainly not all core solutions satisfy both types of monotonicity. On the class of convex games the well known core solution the nucleolus fails to satisfy both coalitional- and population monotonicity, see Hokari (2000a) and Sönmez (1994).

In the present paper we consider the Dutta-Ray solution (or Egalitarian solution, cf. Dutta and Ray (1989)) which on the class of convex games coincides with the Lorenz maximal imputation in the core, see also Hougaard, Peleg and Thorlund-Petersen (2001). As such the Dutta-Ray solution is a core selection rule that explicitly relates to a fairness criterion in terms of equality. Contrary to the nucleolus, it has been shown that the Dutta-Ray solution is indeed population monotonic (Dutta (1990)) and coalitional monotonic (Hokari (2000b)) on the class of convex games. We extend these results by proving that a generalized Lorenz solution concept also satisfies coalitional- as well as population monotonicity.

Subject to the core constraints the generalized Lorenz solution maximizes a social utility function defined as the sum of individual utilities. Each individual utility function is assumed to be strictly concave and it depends only on the agents own payoff (in monetary terms). Therefore, compared to the Dutta-Ray solution (where all agents are supposed to have the same individual utility function) the generalization opens up for asymmetry in terms of individual utilities. Such an asymmetric treatment of players may seem reasonable in cases where, for example, the players differ in wealth prior to the game or, by some other standards, need 'positive' discrimination subject to economic efficiency in terms of the core constraints. Hence, the use of generalized Lorenz solutions follows the same line of argument that underlies the use of weighted Shapley values, asymmetric Nash bargaining solutions etc. (see e.g. Kalai and Samet (1988) and Peters (1992)).

Finally, we notice that Hokari (2002) also introduces a generalization of the Dutta-Ray solution on the class of convex games called a monotone-path Dutta-Ray solution. He claims that our generalized Lorenz solution can be defined as a monotone-path solution. This claim, however, is incorrect and consequently our results below cannot be inferred from his analysis. For example, the monotone-path in his Example 2 iii) is not well defined and his condition iv) is in general not satisfied using our definition. This can be demonstrated by simple examples.

2 Definitions and notation

Assume that there is an infinite set of potential players (agents) indexed by $\mathbf{N}$, the set of natural numbers. Let $\mathcal{N}$ be the class of non-empty, finite subsets of $\mathbf{N}$. A *coalitional game* (with transferable utility) is a pair (N, v), $N \in \mathcal{N}$ where v is a function that associates a real number $v(S)$ with each subset S of N. As usual $v(\emptyset) = 0$. We write v instead of (N, v), when the set of players is fixed and can be omitted for notational simplicity.

Let $\mathbf{R}^N$ denote the set of all functions from N to $\mathbf{R}$. A member x of $\mathbf{R}^N$ is called a payoff vector. If $x \in \mathbf{R}^N$ and $S \subseteq N$ we write $x(S) = \sum_{i \in S} x_i$. Clearly, $x(\emptyset) = 0$.

A game v is *convex* if, for all coalitions S, T

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T).$$

Denote by Γ^N, the class of convex games v for $N \in \mathcal{N}$.

The *core* of v, $C(v)$, is defined by

$$C(v) = \{x \in \mathbf{R}^N \mid x(S) \geq v(S) \text{ for all } S \subset N \text{ and } x(N) = v(N)\}.$$

For $v \in \Gamma^N$, $C(v) \neq \emptyset$ (cf. Shapley 1971).

A *solution* of a game v is a function ϕ which allocates $v(N)$ among all the players, i.e. $\phi(v) = x$ where $x(N) = v(N)$.

Let $u_i : \mathbf{R} \to \mathbf{R},\ i \in \mathbf{N}$ be strictly concave differentiable functions. Now, let $v \in \Gamma^N$ and define the *generalized Lorenz solution* as

$$\phi^{GL}(v) = \arg\max\{\sum_{i \in N} u_i(x_i) \mid x \in C(v)\}.$$

The solution $\phi^{GL}(v)$ is well defined because $u_i,\ i \in N$ are strictly concave and $C(v)$ is compact.

By Theorem 2 in Hougaard, Peleg and Thorlund-Petersen (2001) the generalized Lorenz solution coincides with the *Lorenz-maximal imputation in the core* of a game $v \in \Gamma^N$ if $u_i(x_i) = u(x_i)$, for all $i \in N$. Note also that for convex games the Lorenz-maximal imputation in the core of a game v coincides with the *Egalitarian solution* of Dutta and Ray (1989) – the *Dutta-Ray solution*.

A solution ϕ satisfies *coalitional monotonicity* if for all coalitions S and all $i \in S$ that,

$$[v(S) \leq w(S),\ v(T) = w(T),\ S \neq T] \Rightarrow \phi_i(v) \leq \phi_i(w).$$

Denote by $\mathcal{G} = \{(N, v) | (N, v) \in \Gamma^N\}$ the class of convex games with a finite set of players. Let $x \in \mathbf{R}^N$ and $v \in \Gamma^N$. For $S \subseteq N$, let x^S be the restriction of x to S, and let $v_{|S}$ be the restriction of v to S. Clearly, $v_{|S}$ is convex, i.e. $v_{|S} \in \Gamma^S$.

A solution ϕ satisfies *population monotonicity* if for all $S, N \in \mathcal{N}$ with $S \subset N$, that $\phi_i(v_{|S}) \leq \phi_i(v)$ for all $i \in S$.

3 Coalitional monotonicity

Using the algorithm provided in Dutta and Ray (1989) it is simple to show that the Dutta-Ray solution is coalitional monotonic, see also Hokari (2000b). In this section we will, more surprisingly, demonstrate that the generalized Lorenz solution is coalitional monotonic as well.

Let Γ^N be the set of all convex games on a fixed set of players N, and let $u_i : \mathbf{R} \to \mathbf{R}$ be strictly concave differentiable functions. Consider the generalized Lorenz solution

$$\phi^{GL}(v) = \arg\max\{\sum_{i \in N} u_i(x_i) \mid x \in C(v)\}.$$

First we observe the following result.

Lemma 1 *$\phi^{GL}(v)$ is a continuous function of v.*

Proof: The core $C(v)$ is a continuous correspondence. Hence, by the Theorem of the Maximum (see Mas-Colell, Whinston and Green (1995)) $\phi^{GL}(v)$ is upper hemicontinuous. As ϕ^{GL} is a function it is, in fact, continuous. $\square$

Let $v \in \Gamma^N$ and let $x = \phi^{GL}(v)$. Write $i \sim_x j$ if $u_i'(x_i) = u_j'(x_j)$. Let $\mathcal{P} = \mathcal{P}(x) = \{P_1, \ldots, P_k\}$ be the partition induced by $\sim_x$. We always assume

$$\text{(3.1)} \qquad [i \in P_h, j \in P_l, \text{ and } h < l] \Rightarrow u_i'(x_i) < u_j'(x_j).$$

Thus, $\mathcal{P}$ is actually an ordered partition. We also denote the set of zero-excess coalitions

$$\text{(3.2)} \qquad Z(x) = \{S \subseteq N \mid x(S) = v(S)\}.$$

By Shapley (1971) $Z(x)$ is a ring, that is, it is closed under union and intersection.

Lemma 2 *Let $v \in \Gamma^N$, let $x = \phi^{GL}(v)$, and let $\mathcal{P}(x) = \{P_1, \ldots, P_k\}$ be the partition associated with x. Then $P_1 \cup \ldots \cup P_h \in Z(x)$ for $h = 1, \ldots, k$.*

Proof: Let $i \in P_h$ and $j \in P_l$ where $h < l$. Then $u_i'(x_i) < u_j'(x_j)$. As $x = \phi^{GL}(v)$, there exists $S_{ij} \in Z(x)$ such that $i \in S_{ij}$ and $j \notin S_{ij}$ (no transfer of money from i to j is possible). Let

$$S_i = \bigcap\{S_{ij} \mid j : u_i'(x_i) < u_j'(x_j)\}.$$

Then $S_i \in Z(x)$ and $S_i \subseteq P_1 \cup \ldots \cup P_h$. Thus, for $1 \leq h \leq k$,

$$P_1 \cup \ldots \cup P_h = \bigcup \{S_i \mid i \in P_1 \cup \ldots \cup P_h\} \in Z(x).$$

□

Let $v \in \Gamma^N$, $S \subseteq N, S \neq \emptyset$, and $\bar{t} > 0$. For $0 \leq t \leq \bar{t}$ define

$$(3.3) \qquad w_t(T) = \begin{cases} v(S) + t, & T = S, \\ v(T), & T \neq S. \end{cases}$$

We assume that $w_{\bar{t}} \in \Gamma^N$. Hence, $w_t \in \Gamma^N$ for $0 \leq t \leq \bar{t}$ (by convexity of Γ^N). We shall prove that $\phi_i^{GL}(w_{\bar{t}}) \geq \phi_i^{GL}(v)(= \phi_i^{GL}(w_0))$ for all $i \in S$. First, however, we prove a local version of coalitional monotonicity.

Lemma 3 *Let $v \in \Gamma^N$, let $S \subseteq N, S \neq \emptyset$, let $\bar{t} > 0$ and let $w_{\bar{t}} \in \Gamma^N$ (see (3.3)). Then there exists $0 < \varepsilon \leq \bar{t}$ such that $\phi_i^{GL}(w_t) \geq \phi_i^{GL}(v)$, for all $i \in S$ and $0 \leq t \leq \varepsilon$.*

Proof: Let $y(t) = \phi^{GL}(w_t)$. Thus, $y(0) = \phi^{GL}(w_0) = \phi^{GL}(v)$. If $\sum_{i \in S} y_i(0) > v(S)$, then $S \neq N$ and we may choose $\varepsilon = \sum_{i \in S} y_i(0) - v(S)$. Thus, let

$$(3.4) \qquad \textstyle\sum_{i \in S} y_i(0) = v(S).$$

As $\phi^{GL}(w_t)$ is a continuous function of t, we may choose $0 < \varepsilon \leq \bar{t}$ such that

(3.5) $\quad u_i'(y_i(0)) > u_j'(y_j(0)) \;\Rightarrow\; u_i'(y_i(t)) > u_j'(y_j(t))$, for all $i, j \in N,\; 0 \leq t \leq \varepsilon$;

(3.6) $\quad \sum_{i \in T} y_i(0) > v(T) \;\Rightarrow\; \sum_{i \in T} y_i(t) > v(T),\; T \subseteq N,\; 0 \leq t \leq \bar{\varepsilon}$.

Now, let $0 < t \leq \varepsilon$, let $y(t) = y$, and $y(0) = x$. Denote by

$$Q = \{i \in N \mid y_i < x_i\},$$

the set of players who are worse off in the game w_t. We shall prove that $Q \cap S = \emptyset$. (Thus if $S = N$ then $Q = \emptyset$). Let $\mathcal{P} = (P_1, \ldots, P_k)$ be the ordered partition induced by x, let

$$(3.7) \qquad l = \max\{h \mid Q \cap P_h \neq \emptyset\},$$

and let $\hat{\mathcal{P}} = (\hat{P}_1, \ldots, \hat{P}_q)$ be the partition induced by y. Then, by (3.5), $\hat{\mathcal{P}}$ is a refinement of $\mathcal{P}$, and we shall use this fact in connection with Lemma 2 repeatedly in the following.

Now, if $T' = P_1 \cup \ldots \cup P_l$, then $x(T') = v(T')$, and $y(T') = w_t(T') \geq v(T')$ by Lemma 2. Hence $y(T') \geq x(T')$. Let

$$T = P_1 \cup \ldots \cup P_l \setminus (P_l \cap Q).$$

Clearly, $y(T' \setminus T) < x(T' \setminus T)$, implying that

$$y(T) = y(T') - y(T' \setminus T) > x(T') - x(T' \setminus T) = x(T).$$

Moreover, as $\hat{\mathcal{P}}$ is a refinement of $\mathcal{P}$ (see (3.5)) we have that $P_l \cap Q = \hat{P}_{j_1} \cup \ldots \cup \hat{P}_{j_h}$. Hence, by Lemma 2, $y(T) = w_t(T)$ and by (3.6), $x(T) = v(T)$. Now, we distinguish between two cases:

(i) $S = N$. As $y(T) = w_t(T) > x(T) = v(T)$ we have reached a contradiction (if $Q \neq \emptyset$).

(ii) $S \subseteq N,\ S \neq \emptyset, N$. As $w_t(T) > v(T)$, we obtain that $T = S$. Thus $S \cap Q \cap P_l = \emptyset$. If $l = 1$, then the proof is complete. If $l > 1$, then we will show that $Q \cap P_h = \emptyset$, $h = 1, \ldots, l-1$. Indeed, assume on the contrary that $h < l$ and $Q \cap P_h \neq \emptyset$. Then $Q \cap P_h = \hat{P}_{j_1} \cup \ldots \cup \hat{P}_{j_r}$ by (3.5). Denote $T_* = P_1 \cup \ldots \cup P_h$ and let $T = T_* \setminus (P_h \cap Q)$. Then, by Lemma 2, $y(T) = v(T)$. Hence, by (3.6), $x(T) = v(T)$. Also, clearly, $x(T_*) = v(T_*)$. Thus, $y(T_*) = w_t(T_*) \geq v(T_*) = x(T_*)$. However,

$$y(T_*) = y(T) + y(T_* \setminus T) < x(T) + x(T_* \setminus T) = x(T_*).$$

Thus we have reached the desired contradiction. □

We are now able to prove coalitional monotonicity when the utility functions are differentiable.

Theorem 1 *Let $u_i : \mathbf{R} \to \mathbf{R}$, $i \in N$ be strictly concave differentiable functions. Then $\phi^{GL}(v)$ is coalitional monotonic on Γ^N.*

Proof: Let $v \in \Gamma^N$, let $S \subseteq N,\ S \neq \emptyset$, and let $\bar{t} > 0$. Assume that $w_{\bar{t}}$ is convex (see (3.3)). Let $I = [0, \bar{t}]$. Consider the following subset L of I

$$L = \{t \in I \mid \text{for } 0 \leq \tau \leq t,\ \phi_i^{GL}(w_\tau) \geq \phi_i^{GL}(v) \text{ for all } i \in S\}.$$

L is non-empty ($0 \in L$), and it is closed by Lemma 1. By Lemma 3, L is open. Hence, $L = I$ because I is connected. □

Remark 1: Using the generalized Lorenz solution suitable choice of utility functions u_i may select any imputation in the core. Thus, Theorem 1 indirectly verifies that the core also satisfies coalitional montonicity on the class of convex games. (See e.g. Megiddo (1974) for a precise definition of coalitional monotonicity for set valued functions).

4 Population monotonicity

Proving population monotonicity, we proceed with a number of intermediate lemmata. The first result demonstrates a very useful property of convex games.

Lemma 4 *Let $v \in \Gamma^N$ and $S \subset N$, $x = (x^S, x^{N\setminus S}) \in C(v)$, $y^S \in C(v_{|S})$ and $x(S) = y(S)$. Then $(y^S, x^{N\setminus S}) \in C(v)$.*

Proof: First note that since $x(S) = y(S) = v(S)$, S is a zero-excess coalition at x in v. Assume that there exists a coalition Q that is a zero-excess coalition in v at x with $Q \cap S \neq \emptyset$ and $Q \setminus S \neq \emptyset$. Since Q is zero-excess, $Q \cap S$ is a zero excess coalition in v at x and thereby $y^S(Q \cap S) \geq x^S(Q \cap S)$ since $y^S \in C(v_{|S})$. Therefore

$$v(Q) \leq y^S(Q \cap S) + x^{N\setminus S}(Q \setminus S). \tag{4.1}$$

Now, let $z_t^S = ty^S + (1-t)x^S,\ t \in [0,1]$, and $z_t = (z_t^S, x^{N\setminus S})$. We now show that $z_t \in C(v)$ for all $t \in [0,1]$. Let $L = \{t \in [0,1] \mid z_t \in C(v)\}$. L is nonempty ($0 \in L$) and closed. Moreover, L is open: If $t_0 \in L$ choose $\varepsilon > 0$ such that

$$z_{t_0}(T) > v(T) \Rightarrow z_t(T) > v(T)$$

for all $t_0 - \varepsilon \leq t \leq t_0 + \varepsilon$ for all $T \subseteq N$. As $v(Q) \leq z_t^S(Q \cap S) + x^{N\setminus S}(Q \setminus S)$ if Q is a zero-excess for z_{t_0} by (4.1), we may conclude that $t \in L$ and we get $L = [0,1]$ because $[0,1]$ is connected. □

Corollary 1 *Let $x = \phi^{GL}(v)$. For $S \subset N$ if $x(S) = v(S)$ then $x^S = \phi^{GL}(v_{|S})$.*

Proof: It is clear that $x^S \in C(v_{|S})$. Assume that $x^S \neq \phi^{GL}(v_{|S}) = y^S$. Then $\sum_{i\in S} u_i(x_i) < \sum_{i\in S} u_i(y_i)$. By Lemma 4, $y^S, x^{N\setminus S}) \in C(v)$. Since $\sum_{i\in S} u_i(x_i) + \sum_{i\in N\setminus S} u_i(x_i) < \sum_{i\in S} u_i(y_i^S) + \sum_{i\in N\setminus S} u_i(x_i)$, it is a contradiction that $x = \phi^{GL}(v)$. □

For $n \in N$ let $\phi^c(v) = \arg\max\{\sum_{i\in N} u_i(x_i) + cx_n \mid x \in C(v)\}$, $c \in \mathbf{R}$.

Lemma 5 *$\phi^c(v)$ is a continuous function of c.*

Proof: Let c_k, $k = 1, 2, 3...$, be a convergent sequence, $c_k \to c$. Assume that $\phi^{c_k}(v) \not\to \phi^c(v)$. Since $C(v)$ is compact, there exists a subsequence $c_{k(p)}$, $p = 1, 2, 3...$, where $\phi^{c_{k(p)}}(v)$ is convergent, yet $\phi^{c_{k(p)}}(v) \not\to \phi^c(v)$. Let $\phi^{c_{k(p)}}(v) = x^{k(p)} \to x$, and $\phi^c(v) = y$. Note that $\sum_{i\in N} u_i(y_i) + cy_n - \sum_{i\in N} u_i(x_i) - cx_n > 0$. Since

$$\lim_{p\to\infty} \left(\sum_{i\in N} u_i(x_i) + cx_n - \sum_{i\in N} u_i(x_i^{k(p)}) - c_{k(p)} x_n^{k(p)} \right) = 0,$$

and $\lim_{p\to\infty} \left(\sum_{i\in N} u_i(y_i) + cy_n - \sum_{i\in N} u_i(y_i) - c_{k(p)} y_n \right) = 0$, for p sufficiently large ($c_{k(p)}$ sufficiently close to c), $\sum_{i\in N} u_i(y_i) + c_{k(p)} y_n - \sum_{i\in N} u_i(x_i^{k(p)}) - c_{k(p)} x_n^{k(p)} > 0$, contradicting that $x^{k(p)} = \arg\max\{\sum_{i\in N} u_i(x_i) + c_{k(p)} x_n \mid x \in C(v)\}$. □

Lemma 6 *If c is sufficiently high, $\phi_n^c(v) = v(N) - v(N\setminus n)$.*

Proof: Since $C(v)$ is compact, and u_i is differentiable on $\mathbf{R}$ for all i, $\sup\{\sum_{i\in N} u_i'(x_i) \mid x \in C(v)\}$ is finite. On the other hand, $cx_n - cy_n$, $x_n > y_n$, goes towards infinity as $c \to \infty$. It is thereby clear that if c is sufficiently large then $\phi_n^c(v) = \max\{x_n \mid x \in C(v)\}$. By Shapley's (1971) completeness characterization of the core this solution is given by $v(N) - v(N\setminus n)$. □

Corollary 2 *If c is sufficiently high, $\phi^c(v) = \left(\phi^{GL}(v_{|N\setminus n}), v(N) - v(N\setminus n)\right)$.*

Proof: Follows directly from Lemma 6 and Corollary 1. □

Lemma 7 *Let $c > d$. Then $\phi_n^c(v) \geq \phi_n^d(v)$.*

Proof: Let $x = \phi^c(v)$ and let $y = \phi^d(v)$. Now assume that $x_n < y_n$. Since

$$\sum_{i \in N} u_i(x_i) + dx_n < \sum_{i \in N} u_i(y_i) + dy_n,$$

and since $c > d$, then also

$$\sum_{i \in N} u_i(x_i) + cx_n < \sum_{i \in N} u_i(y_i) + cy_n,$$

contradicting that $x = \phi^c(v)$. □

Note that Lemmata 1,5 and 7 do not hinge on convexity and apply to all balanced games v.

Lemma 8 *There exists $\varepsilon > 0$ such that $\phi_i^c(v) \leq \phi_i^{c-t}(v)$ for $i \in N \setminus n$ and $0 \leq t \leq \varepsilon$.*

Proof: Let $x = \phi^c(v)$ and let $y(t) = \phi^{c-t}(v)$. Let $\mathcal{P} = \{P_1, ..., P_k\}$ be the ordered partition induced by x. As $\phi^c(v)$ is a continuous function of c, we may, by differentiability of u_i for $i \in N$, choose ε such that

$$u_i'(x_i) > u_j'(x_j) \Rightarrow u_i'(y_i(t)) > u_j'(y_j(t)),$$

for all $i, j \in N, 0 \leq t \leq \varepsilon$.

Now we shall prove that $\phi_i^c(v) \leq \phi_i^{c-t}(v)$ for $i \in N \setminus n$, $0 \leq t \leq \varepsilon$. Let $y(t) = y$, and let $\widehat{\mathcal{P}} = \{\widehat{P}_1, ..., \widehat{P}_q\}$ be the partition induced by y. Note that $\widehat{\mathcal{P}}$ is a refinement of $\mathcal{P}$.

Let $n \in P_l$. The coalition $\bigcup_{h=1}^{l-1} P_h$ is a zero-excess coalition at x and at y. Thus, by Corollary 1, $x_i = y_i$ for $i \in \bigcup_{h=1}^{l-1} P_h$. Moreover, we shall prove that $x_i = y_i$ for $i \in \bigcup_{h=l+1}^{k} P_h$. Assume that $x_i \neq y_i$ for some $i \in \bigcup_{h=l+1}^{k} P_h$. Let $\phi^c(v) = (x^{\cup_{h=1}^{l} P_h}, x^{\cup_{h=l+1}^{k} P_h}), \phi^{c-t}(v) = (y^{\cup_{h=1}^{l} P_h}, y^{\cup_{h=l+1}^{k} P_h})$ where $x^{\cup_{h=l+1}^{k} P_h} \neq y^{\cup_{h=l+1}^{k} P_h}$. By Lemma 4, $(x^{\cup_{h=1}^{l} P_h}, y^{\cup_{h=l+1}^{k} P_h}) \in C(v)$. Thus $\sum_{i \in \cup_{h=l+1}^{k} P_h} u_i(x_i) > \sum_{i \in \cup_{h=l+1}^{k} P_h} u_i(y_i)$ since $x = \phi^c(v)$. Furthermore, by Lemma 4, $(y^{\cup_{h=1}^{l} P_h}, x^{\cup_{h=l+1}^{k} P_h}) \in C(v)$. Thus

$$\sum_{i \in \cup_{h=l+1}^{k} P_h} u_i(x_i) < \sum_{i \in \cup_{h=l+1}^{k} P_h} u_i(y_i)$$

since $y = \phi^{c-t}(v)$ and a contradiction is obtained. To sum up, $x_i = y_i$ for $i \notin P_l$.

Consider now the coalition P_l. Let $\{\widehat{P}_{j_1}, ..., \widehat{P}_{j_r}\}$ be the partition of P_l induced by y. Assume for some player $i' \in P_l, i' \neq n$, that $x_{i'} > y_{i'}$. Let $i' \in \widehat{P}_{j_m}$. Now consider a player $i'' \in \bigcup_{h=m}^{r} \widehat{P}_{j_h}$ such that $x_{i''} \neq y_{i''}$. If $i'' = n$ then, by Lemma 7, $x_{i''} > y_{i''}$. If $i'' \neq n$ then $x_{i''} > y_{i''}$ since $u'_{i'}(x_{i'}) = u'_{i''}(x_{i''})$ and $u'_{i'}(y_{i'}) \leq u'_{i''}(y_{i''})$. Therefore $x_i > y_i$ for all $i \in \bigcup_{h=m}^{r} \widehat{P}_{j_h}$. Thus $m \geq 2$, and $x(\bigcup_{h=1}^{m-1} \widehat{P}_{j_h}) < y(\bigcup_{h=1}^{m-1} \widehat{P}_{j_h})$ contradicting that $\{\bigcup_{h=1}^{l} P_h \setminus \bigcup_{h=m}^{r} \widehat{P}_{j_h}\}$ is a zero-excess coalition at y. □

Lemma 9 *Let $c > d$. Then $\phi_i^c(v) \leq \phi_i^d(v)$ for all $i \in N \setminus n$.*

Proof: Let $x = \phi^c(v)$ and let $y = \phi^d(v)$. Let $I = [d, c]$. Consider the following subset of I

$$L = \{t \in I \mid t \leq t' \leq c, \phi_i^c(v) \leq \phi_i^{t'}(v), i \in N \setminus n\}.$$

L is non-empty ($c \in L$), and it is closed by Lemma 5. By Lemma 8, L is open. Hence, $L = I$ because I is connected. □

With the preceding lemmata in place, we are able to prove that the generalized Lorenz solution satisfies population monotonicity when the utility functions are differentiable.

Theorem 2 *The generalized Lorenz solution $\phi^{GL}(v) = \arg\max\{\sum_{i \in N} u_i(x_i) \mid x \in C(v)\}$ satisfies population monotonicity on the class of convex games $\mathcal{G}$.*

Proof: Let $v \in \Gamma^N$. We shall verify that $\phi_i^{GL}(v_{|N\setminus n}) \leq \phi_i^{GL}(v)$ for all $i \in N \setminus n$. By Corollary 2, there exists $\overline{c} \geq 0$ such that $\phi_i^{GL}(v_{|N\setminus n}) = \phi_i^{\overline{c}}(v)$ for all $i \in N \setminus n$. By Lemma 9, $\phi_i^{\overline{c}}(v) \leq \phi_i^0(v) = \phi_i^{GL}(v)$ for all $i \in N \setminus n$ and we are done. □

5 Extensions

Theorem 1 has the following bounded version. Let $a, b \in \mathbf{R}$ and let

$$\Gamma^N(a, b) = \{v \in \Gamma^N \mid a \leq v(i),\ i \in N \text{ and } v(N) \leq b\}.$$

Theorem 3 *Let $u_i : [a, b^*] \to \mathbf{R}$, $i \in N$, where $b^* = b - (n-1)a$, be strictly concave and differentiable functions. Then $\phi^{GL}(v)$ is continuous and coalitional monotonic on $\Gamma^N(a, b)$.*

The proof of Theorem 3 is similar to that of Theorem 1. The following result is an important generalization of Theorem 1 as it states that coalitional monotonicity of the generalized Lorenz solution does not rest on differentiability of the individual utility functions.

Theorem 4 *Let $u_i : \mathbf{R} \to \mathbf{R}$, $i \in N$ be strictly concave functions. For $v \in \Gamma^N$ define $\phi(v) = \arg\max\{\sum_{i \in N} u_i(x_i) \mid x \in C(v)\}$. Then $\phi(v)$ is continuous and coalitional monotonic.*

Clearly, we only have to prove coalitional monotonicity. Also, it is sufficient to prove coalitional monotonicity on $\Gamma^N(a, b)$ for arbitrary a, b. The proof in this case relies on the following result.

Lemma 10 *Let $u : [a, b] \to \mathbf{R}$, be a continuous concave function. Then there exists a sequence $u^k : [a, b] \to \mathbf{R}$, $k = 1, 2, \ldots$, such that u^k is strictly concave and differentiable, and $u^k \to u$ uniformly on $[a, b]$.*

The proof of Lemma 10 is left to the reader.

We shall now prove Theorem 4.

Proof (of Theorem 4): Let $v \in \Gamma^N(a, b)$, let $t > 0$, let $S \subseteq N$, $S \neq \emptyset$, and let $w_t \in \Gamma^N(a, b)$ (see (3.3)). For each $i \in N$ we choose a sequence of strictly concave and differentiable functions u_i^k on $[a, b^*]$, $k = 1, 2, \ldots$, such that $u_i^k \to u_i$ uniformly on $[a, b^*]$. Let

$$\phi^k(v_*) = \arg\max\{\sum_{i \in N} u_i^k(x_i) \mid x \in C(v_*)\}, \quad v_* \in \Gamma^N(a, b).$$

Then $\phi^k(v) \to \phi(v)$, $\phi^k(w_t) \to \phi(w_t)$, and $\phi_i^k(v) \leq \phi_i^k(w_t)$ for all $i \in S$ (by Theorem 3). Hence, $\phi_i(v) \leq \phi_i(w_t)$ for all $i \in S$. □

Now turning towards an extension of Theorem 2, let $a, b \in \mathbf{R}$. Denote by $\mathcal{G}(a, b)$ the bounded class of convex games

$$\mathcal{G}(a, b) = \left\{(N, v) \mid (N, v) \in \Gamma^N(a, b)\right\}.$$

As in the case of Theorem 3 we can obtain a bounded version of Theorem 2.

Theorem 5 *Let $u_i : [a, b^*] \to \mathbf{R}$, for all i, where $b^* = b - (n-1)a$, be strictly concave and differentiable functions. The generalized Lorenz solution $\phi^{GL}(v) = \arg\max\{\sum_{i \in N} u_i(x_i) \mid x \in C(v)\}$ satisfies population monotonicity on $\mathcal{G}(a, b)$.*

Proof: Since $u_i'(a) < \infty$, $u_i'(b^*) > -\infty$ there exists a strictly concave and differentiable extension $\overline{u}_i : \mathbf{R} \to \mathbf{R}$ of u_i for all i. Replace u_i by $\overline{u}_i$ for all i and apply Theorem 2. $\square$

We are then able to prove that Theorem 2 does not hinge on the differentiability of the individual utility functions.

Theorem 6 *Let $u_i : \mathbf{R} \to \mathbf{R}$ be strictly concave functions for all i. The generalized Lorenz solution $\phi^{GL}(v) = \arg\max\{\sum_{i \in N} u_i(x_i) \mid x \in C(v)\}$ satisfies population monotonicity on the class of convex games $\mathcal{G}$.*

Proof: It is sufficient to prove population monotonicity on $\mathcal{G}(a, b)$ for arbitrary a, b. Let $v \in \Gamma^N(a, b)$. We shall verify that $\phi_i^{GL}(v_{|N \backslash n}) \leq \phi_i^{GL}(v)$ for all $i \in N \backslash n$. For each $i \in N$ we can choose a sequence of strictly concave and differentiable functions u_i^k on $[a, b^*]$, $k = 1, 2, ...$, such that $u_i^k \to u_i$ uniformly on $[a, b^*]$. Let

$$\phi^k(v_*) = \arg\max\{\sum_{i \in N} u_i^k(x_i) \mid x \in C(v_*)\}, v_* \in \Gamma^N(a, b).$$

Then $\phi^k(v) \to \phi^{GL}(v)$, $\phi^k(v_{|N \backslash n}) \to \phi^{GL}(v_{|N \backslash n})$, and $\phi_i^k(v_{|N \backslash n}) \leq \phi_i^k(v)$ for all $i \in N \backslash n$ (by Theorem 5). Hence, $\phi_i^{GL}(v_{|N \backslash n}) \leq \phi_i^{GL}(v)$ for all $i \in N \backslash n$. $\square$

6 Counter examples

To demonstrate that separability of the social welfare function is essential for Theorems 1-6, we provide the following example:

Example 1: Let $N = \{1, 2, 3\}$, and consider the following convex game v: $v(i) = 0$, for $i = 1, 2, 3$, $v(1, 3) = v(2, 3) = 1/3$, $v(1, 2) = \varepsilon$, $0 < \varepsilon < 1/3$, and $v(N) = 1$. Likewise, define another convex game w as $w(1, 2) = 1/3 > v(1, 2)$ and $w(S) = v(S)$ otherwise.

Let $x = (x_1, x_2, x_3)$ be a vector of payoffs and let $\phi(v) = \arg\max\{u(x) \mid x \in C(v)\}$ be a solution w.r.t. the following social welfare function

$$u(x) = \sqrt{x_1 + \alpha x_3 + \beta} + \sqrt{x_2 + \gamma} + \delta\sqrt{x_3},$$

where $\alpha, \beta, \gamma, \delta > 0$. Choose α, β, γ such that

$$\frac{1}{\sqrt{\alpha + \beta}} < \frac{1}{\sqrt{\gamma}} \quad \text{and} \quad \frac{1}{\sqrt{1/3 + 2\alpha/3 + \beta}} > \frac{1}{\sqrt{\gamma}},$$

implying that $\alpha > 1$. Then, there exists $\varepsilon > 0$ sufficiently small such that

$$\frac{1}{\sqrt{\alpha(1-\varepsilon) + \beta}} < \frac{1}{\sqrt{\varepsilon + \gamma}}.$$

That is $\phi(v) = (0, \varepsilon, 1 - \varepsilon)$ whereas $\phi(w) = (1/3, 0, 2/3)$ contradicting coalitional monotonicity. Moreover, $\phi_1(v_{|\{1,2\}}) > 0 = \phi_1(v)$ contradicting population monotonicity. □

To demonstrate that convexity is essential for Theorem 1-6, we provide the following example (note that in Young (1985) the counter example concerns a five player game; in Housman and Clark (1998) a four player game):

Example 2: Let $N = \{1, 2, 3\}$ and consider the following totally balanced game: $v(1) = v(2) = 0$, $v(3) = 2$, $v(1,2) = v(1,3) = 2$, and $v(2,3) = v(N) = 4$. Clearly, the game v is not convex as for instance $v(1,2) + v(2,3) > v(N) + v(2)$. Now, let $x = (x_1, x_2, x_3)$ be a pay-off vector and let $\phi^{GL}(v) = \arg\max\{u(x) \mid x \in C(v)\}$ be the generalized Lorenz solution w.r.t. the following social welfare function

$$u(x) = \sqrt{x_1} + 2\sqrt{x_2} + 10\sqrt{x_3}.$$

We find that $\phi^{GL}(v) = (0, 2, 2)$.

Now, define a new (non-convex) game $w(N) = 5 > v(N)$, and $w(S) = v(S)$ otherwise. Here $\phi^{GL}(w) = (2/5, 8/5, 3)$ contradicting coalitional monotonicity. Furthermore, $\phi_3^{GL}(v_{|\{2,3\}}) > 2 = \phi_3^{GL}(v)$ contradicting population monotonicity □

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